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question requiring attentive consideration whether the additions to be made at the present time should be of the full amount of 1½ per cent. On the other hand, if $L+R>B$, the assets of the Office, present and prospective, are more than sufficient to make the proposed additions; and if the excess is considerable, it may be prudent to declare a bonus at the present time at a higher rate than 1½ per cent. In this process no account has been taken of the part of the premium added to provide for expenses; this will render some modification of the above necessary in practice.

On the Use of the Integral Calculus in determining Averages, with certain Applications to the Theory of Life Contingencies. By SAMUEL YOUNGER, *of the Engineers' Life Office.*

IN a paper* which was read before the Institute of Actuaries on the 8th January, 1855, the idea was first suggested by Mr. Farren of introducing into the theory of life contingencies the hypothesis of a constantly fluctuating rate of interest, in lieu of the supposition hitherto adopted of a uniform rate.

Abundant reasons are assigned, in the paper referred to, for making the proposed change; and no one, at all acquainted with the practice of life assurance, can fail to see that the effect of Mr. Farren's suggestion, if carried out, will be to base all computations upon an assumption which is in the closest possible accordance with actual facts.

That opposition, in some quarters, will be displayed to any innovation upon existing methods and formulæ, is tolerably certain. It will be maintained, perhaps, that while the proposed amendment will not very materially alter numerical results obtainable by the present mode of calculation, it will entail much additional trouble in finding them; that the analytical formulæ will be too complex to be of practical utility; that, in short, the whole existing fabric will be destroyed, and every actuary compelled to begin his studies again. The main object of the present paper is to show the fallacy of these objections, by exhibiting the results which flow from the assumption of an ever varying rate of interest—first, in the case of life annuities, and then in the case of life assurances. It will be seen, that if a new table of annuities on lives be calculated by the formula here given, all the general results, which have been arrived at on the supposition of a uniform rate of interest (with certain exceptions, which are specified), will remain unaltered and will hold good, when the rate is liable to constant fluctuation.

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* *On the Improvement of Life Contingency Calculation.* By Edwin James Farren, Esq. See *Assurance Magazine*, vol. v., pp. 185, &c.

It may be as well to explain at the outset how the integral calculus can be made to afford the average value of a function between given limits.

Let $\phi(x)$ be any function which is continuous and finite for all values of x from a to $a+n$, and let dx be the infinitesimal difference between any two successive values of the variable; then, if k be an indefinitely great number such that $k.dx=n$, we have

$$\int_a^{a+n}\phi(x)dx=\left\{\phi(a)+\phi(a+dx)+\phi(a+2dx)\ \ldots\ +\phi(a+\overline{k-1}\,.\,dx)\right\}^{*}dx\,;$$

or, since $dx=\frac{n}{k}$, the integral is

$$=\frac{n}{k}\left\{\phi(a)+\phi(a+dx)+\phi(a+2dx)\ \ldots\ +\phi(a+\overline{k-1}\,.\,dx)\right\};$$

$$\therefore\frac{1}{n}\int_a^{a+n}\phi(x)dx=\frac{1}{k}\left\{\phi(a)+\phi(a+dx)+\phi(a+2dx)\ \ldots\ +\phi(a+\overline{k-1}\,.\,dx)\right\}.$$

Now the right-hand side of this equation denotes the kth part of k quantities, and therefore expresses the average of them all; but the quantities themselves comprise every possible value of $\phi(x)$ between the given limits of x (k being indefinitely great): hence

$$\frac{1}{n}\int_a^{a+n}\phi(x)dx=\text{average value of }\phi(x)\text{ from }x=a\text{ to }x=a+n.$$

In general, therefore, if $\int\phi(x)=\psi(x)+C$, the average value of $\phi(x)$, between the limits α and β of x, is

$$=\frac{\psi(\beta)-\psi(\alpha)}{\beta-\alpha}.$$

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[*] The series stops at the term $\phi(a+\overline{k-1}\,.\,dx)$, or, which is the same thing, at $\phi(a+n-dx)$. It is well known that $\int_a^b\phi(x)dx$ denotes the sum of all the values of $\phi(x)dx$ from $x=a$, *up to* but *not including* $x=b$; in other words, the last value of x is $b-dx$. It is sufficiently evident, from the following known property of a definite integral, that *both* limits cannot be contained. Let $a_1\,a_2\,a_3\ldots\,a_n$ be a series of quantities in order of magnitude, a_1 being the least and a_n the greatest, then

$$\int_{a_1}^{a_n}\phi(x)dx=\int_{a_1}^{a_2}\phi(x)dx+\int_{a_2}^{a_3}\phi(x)dx\ldots\ldots+\int_{a_{n-1}}^{a_n}\phi(x)dx.$$

If both limits were included, the last value of x in the series symbolized by $\int_{a_1}^{a_2}\phi(x)dx$ would be the same as the first value in $\int_{a_2}^{a_3}\phi(x)dx$, and so on for the rest; thus many values of the variable would be repeated twice, and the property could not hold.

We might have arrived at the same result by the theory of probabilities, since every question of average is one of probability, and *vice versâ.* Thus, let $\phi(x)$ be the chance that an event will occur which can only happen by x having some value between α and β, and suppose all the values between these limits to be equally likely *à priori,* then the probability that x will have any one of the values, to the exclusion of the rest, is $\frac{dx}{\beta-\alpha}$, and the chance of the event occurring is

$$=\frac{dx}{\beta-\alpha}\phi(x) \therefore \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\phi(x)dx=\text{the whole probability};$$

and this must, by the nature of the case, be the average value of $\phi(x)$.

As a first application of the principle now laid down, let it be required to find the present value of £1 due at the end of n years. If the rate of interest were uniformly x per £, the answer would be $(1+x)^{-n}$: hence, when the rate fluctuates between α and β, the present value, or

$$v_n=\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}(1+x)^{-n}.dx$$

$$=\frac{1}{(n-1)(\beta-\alpha)}\left\{\frac{1}{(1+\alpha)^{n-1}}-\frac{1}{(1+\beta)^{n-1}}\right\} \quad . \quad . \quad (1)$$

When $n=1$, the expression assumes the indeterminate form $\frac{o}{o}$. Its value in this particular case, by the ordinary method of valuing a vanishing fraction, is

$$v_1=\frac{1}{\beta-\alpha}\log_e\left(\frac{1+\beta}{1+\alpha}\right) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If in the formula (1) we write $-n$ for n, the result will be the *amount* of £1 in n years,

$$=\frac{1}{(n+1)(\beta-\alpha)}\left\{(1+\beta)^{n+1}-(1+\alpha)^{n+1}\right\}.$$

From (1) and (2) we may deduce the present value of a life annuity. Let $p_{m(n)}$ denote the probability that a person aged m will live to the age of $m+n$ years; then, putting a'_m for the value of an annuity of £1 on the life when the rate of interest fluctuates from α to β, we have

$$a'_m=\Sigma p_{m(n)}\,.\,v_n$$

$$=\frac{p_{m(1)}}{\beta-\alpha}\log_e\left(\frac{1+\beta}{1+\alpha}\right)+\Sigma p_{m(n+1)}\,.\,v_{n+1} \quad . \quad . \quad (3)$$

where Σ indicates the sum of all the values of the quantity before which it stands, when n therein is made successively equal to 1, 2, 3 . . . . . $(z-m-1)$, the last age in the table of mortality being z. Now, by (1),

$$\Sigma p_{m(n+1)} \,.\, v_{n+1} = \frac{1}{\beta-a}\,\Sigma\left\{\frac{p_{m(n+1)}}{n(1+a)^n} - \frac{p_{m(n+1)}}{n(1+\beta)^n}\right\}$$

$$= \frac{rr'}{r-r'}\left(\Sigma\frac{1}{n}p_{m(n+1)} \,.\, r^n - \Sigma\frac{1}{n}p_{m(n+1)} \,.\, r'^n\right)$$

by putting $\frac{1}{1+a}=r$ and $\frac{1}{1+\beta}=r'$.

Substituting this last result in (3), the solution of the problem will be, theoretically, complete: it would, however, be extremely inconvenient to frame an annuity table from such a formula, on account of the labour involved in the calculation of $\Sigma\frac{r^n}{n}p_{m(n+1)}$ and $\Sigma\frac{r'^n}{n}p_{m(n+1)}$. Instead, therefore, of proceeding further with this solution, I will deduce the same result by a more connected process, and then show how it may be thrown into a form better adapted for practical use.

Using a_m to denote the value of an annuity when the rate of interest is uniform and equal to x per £, and a'_m to denote the same when the rate fluctuates between i and i', we have

$$a_m = \phi(x) = \frac{p_{m(1)}}{1+x} + \frac{p_{m(2)}}{(1+x)^2} + \frac{p_{m(3)}}{(1+x)^3} +, \text{\&c.}$$

$$\therefore\ a'_m = \frac{1}{i'-i}\int_i^{i'} \phi(x)dx$$

$$= \frac{1}{i'-i}\left\{p_{m(1)} \log_e\left(\frac{1+i'}{1+i}\right) + p_{m(2)}\left(\frac{1}{1+i} - \frac{1}{1+i'}\right) + \tfrac{1}{2}p_{m(3)}\left(\frac{1}{\overline{1+i}|^2} - \frac{1}{\overline{1+i'}|^2}\right) +, \text{\&c.}\right\}$$

$$= \frac{rr'}{r-r'}\left\{p_{m(1)} \log_e \frac{r}{r'} + \Sigma\frac{1}{n}p_{m(n+1)}r^n - \Sigma\frac{1}{n}p_{m(n+1)}r'^n\right\} \quad . \quad (4)$$

Thus far we had arrived in the former solution. Now it will be found desirable to express the final result in terms of two quantities, Q and Q′, such that

$$Q_m = \Sigma\frac{1}{n}\,p_{m(n)} \,.\, r^n = p_{m(1)} \,.\, r + \frac{1}{2}p_{m(2)} \,.\, r^2 + \frac{1}{3}p_{m(3)} \,.\, r^3 +, \text{\&c.}$$

$$Q'_m = \Sigma\frac{1}{n}\,p_{m(n)}r'^n = p_{m(1)} \,.\, r' + \frac{1}{2}\,p_{m(2)} \,.\, r'^2 + \frac{1}{3}p_{m(3)} \,.\, r'^3 +, \text{\&c.}$$

From the obvious identity

$$p_{m(n+1)} = p_{m(1)} \times p_{m+1(n)}$$

we get $\Sigma\frac{1}{n}p_{m(n+1)} \,.\, r^n = p_{m(1)}\,\Sigma\frac{1}{n}p_{m+1(n)} \,.\, r^n = p_{m(1)} \,.\, Q_{m+1}$;

hence, by (4),

$$a'_m = \frac{rr'}{r-r'} p_{m(1)} \left(\log_e \frac{r}{r'} + Q_{m+1} - Q'_{m+1} \right) \quad : \quad . \quad . \quad (5)$$

To allow for the occasional unproductiveness of money, through delay in its investment and other causes, it would no doubt be advisable in practice to take $i=o$ or $r=1$: in this case, (5) becomes

$$a'_m = \frac{r'}{1-r'} p_{m(1)} \left(\log_e \frac{1}{r'} + \Sigma \frac{1}{n} p_{m+1(n)} - Q'_{m+1} \right) \quad . \quad . \quad (6)$$

To render the last two results fit for use, it is only necessary to show how the Q functions may be calculated. Jones's D column would certainly answer the purpose, since

$$Q_m = \frac{1}{D_m} \left(D_{m+1} + \frac{1}{2} D_{m+2} + \frac{1}{3} D_{m+3} +, \text{\&c.} \right),$$

but the process would be very tedious, particularly for the smaller values of *m*; and to this method, moreover, there is the great objection that the value of Q, when found for any one age, will be of no service in determining its value for any other age. The difficulty, however, is easily removed, for we have

$$Q_{m+1} = \frac{1}{D_{m+1}} \left(D_{m+2} + \frac{1}{2} D_{m+3} + \frac{1}{3} D_{m+4} +, \text{\&c.} \right);$$

$$\therefore Q_m D_m - Q_{m+1} D_{m+1} = D_{m+1} - \left(1 - \frac{1}{2}\right) D_{m+2} - \left(\frac{1}{2} - \frac{1}{3}\right) D_{m+3} - \left(\frac{1}{3} - \frac{1}{4}\right) D_{m+4} -, \text{\&c.}$$

$$\therefore Q_m = \frac{1}{D_m} \left\{ (1 + Q_{m+1}) D_{m+1} - \left(\frac{1}{2} D_{m+2} + \frac{1}{2 \cdot 3} D_{m+3} + \frac{1}{3 \cdot 4} D_{m+4} +, \text{\&c.} \right) \right\} \quad . \quad (7)$$

The latter series, by virtue of the factor $\frac{1}{D_m}$, with which it is connected, converges so rapidly, that in most cases the first six or eight of its terms will give the sum with all needful accuracy. By commencing therefore with the highest value of *m* in the table of mortality (for which Q_m of course $=o$), we get the value of Q for the next less age; and so, working backwards with the ages, a Q table is readily formed.

When a table of annuity values has to be calculated from (6), the quantity $\Sigma \frac{1}{n} p_{m+1(n)}$ or $(Q_{m+1})_{i=o}$ for the different ages may be handled in a manner precisely analogous to the above, for when

$i=o$, $\qquad Q_{m+1} = \Sigma \frac{1}{n} p_{m+1(n)}$, and $D_m = l_m$;

therefore, by (7),

$$\Sigma \frac{1}{n} p_{m(n)} = \frac{1}{l_m} \left\{ \left(1 + \Sigma \frac{1}{n} p_{m+1(n)} \right) l_{m+1} - \left(\frac{1}{2} l_{m+2} + \frac{1}{2 \cdot 3} l_{m+3} + \frac{1}{3 \cdot 4} l_{m+3} +, \text{\&c.} \right) \right\} \quad (8)$$

To pass on to the case of life assurances. Let it be required to find the value of £1, payable at the death of a person aged m years, or, rather, at the end of the year in which the life may fail. When the rate of interest x is constant, the value is denoted by A_m; and when x is liable to fluctuate between i and i', let the same be expressed by A'_m: then

$$A_m = \frac{1-p_{m(1)}}{1+x} + \frac{p_{m(1)}-p_{m(2)}}{(1+x)^2} + \frac{p_{m(2)}-p_{m(3)}}{(1+x)^3} +, \&c.$$

Hence, multiplying by $\frac{dx}{i-i'}$, and integrating, as in the case of an annuity, from $x=i$ to $x=i'$, there results

$$A'_m = \frac{rr'}{r-r'}\left\{(1-p_{m(1)})\log_e\frac{r}{r'} + (p_{m(1)}-p_{m(2)})(r-r') + \frac{1}{2}(p_{m(2)}-p_{m(3)})(r^2-r'^2) +, \&c.\right\}$$

$$= \frac{rr'}{r-r'}\left\{\log_e\frac{r}{r'} + p_{m(1)}(r-r') + \frac{1}{2}p_{m(2)}(r^2-r'^2) +, \&c.\right\} - a'_m$$

$$= \frac{rr'}{r-r'}\left(\log_e\frac{r}{r'} + Q_m - Q'_m\right) - a'_m$$

Now (5) gives $\frac{rr'}{r-r'}\left(\log_e\frac{r}{r'} + Q_m - Q'_m\right) = \frac{a'_{m-1}}{p_{m-1(1)}}$

$$\therefore A'_m = \frac{a'_{m-1}}{p_{m-1(1)}} - a'_m \quad . \quad . \quad . \quad (9)$$

This solution is merely given as an example of the manner in which such questions might be worked. The result may be deduced far more easily, thus. The common expression for the single premium, when the rate of interest is x, is $\frac{1-xa_m}{1+x}$; this may be written $\frac{a_{m-1}}{p_{m-1(1)}} - a_m$. Now, remembering that a_{m-1} and a_m are functions of x, and that the average value of the latter expression is A'_m, we have

$$A'_m = \frac{1}{(i'-i)p_{m-1(1)}}\int_i^{i'} a_{m-1}.dx - \frac{1}{i'-i}\int_i^{i'} a_m.dx;$$

but

$$\frac{1}{i'-i}\int_i^{i'} a_m.dx = a'_m,$$

$$\therefore A'_m = \frac{a'_{m-1}}{p_{m-1(1)}} - a'_m,$$

the same result as before. The annual premium, of course, $= \frac{A'_m}{1+a'_m}$.

It is unnecessary to deal further with particular questions, the general rule to be followed being sufficiently obvious; for suppose that we have any formula, deduced on the hypothesis of a uniform rate of interest, and involving annuity values and probabilities *only* (*i. e.*, one in which i is not *visible*), and we wish to pass to the corresponding result, when interest is variable, it is only necessary to accent the a's and the transformation is complete: in short (with reference to actual practice), we have only to use our present formulæ, and employ a table of annuities calculated by either (5) or (6).

When an expression such as $\frac{1-i.a_m}{1+i}$ is met with, in which i appears, it must be thrown into some equivalent formula in which i is not found, and then the above rule will at once apply.

All that has been advanced with reference to a single life, is applicable to any number of joint lives—that is to say, the process in both cases is the same, and the results are precisely analogous in form.

Hitherto it has been assumed that the amount assured is a fixed sum; but the problem would be virtually the same if that amount were liable to fluctuation between given limits, as well as the rate of interest. To take an example:—It has been happily suggested (I do not know by whom, originally), that to meet the requirements of certain cases which sometimes occur in practice, the amount of a policy might be expressed in Consols. Now, although the price of the stock when the policy will become a claim must be a matter of uncertainty, and the Company's precise risk consequently unknown, still we might with sufficient safety assign two limits between which it could be assumed, without material error, the price of the stock would *certainly* be at the period in question; such limits, however, must be so chosen as to allow of the further assumption that the lower is as likely to be the price as the higher, and that any intermediate price is as probable as either of the extremes. It would then only be requisite to convert the stock into sterling, at a quotation midway between the assigned limits, and to demand the corresponding premium. The effect of variable interest would be also taken into account, if the new table of annuities were used in the calculation of the premium.

In closing this brief paper, it may be as well to state—inasmuch as the theory of the subject has been alone touched upon here—that I propose, in a supplementary communication, to give a table of life annuities, framed upon the formula (6), together with a few other tabular results, and some examples of their application.